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1-EDGE BALANCE INDEX SETS OF $C_n \times P_3$ AND $K_{n,n}$

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Abstract

Let G be a graph with vertex set V, edge set E and $Z_2 = \{0, 1\}$. Let f be a labeling from E to Z_2, so that the labels of the edges are 0 or 1. The edges labelled 1 are called 1-edges and edges labelled 0 are called 0-edges. The edge labeling f induces a vertex labeling $f^* : V \longrightarrow Z_2$ defined by

$$f^*(v) = \begin{cases} 1 & \text{if the number of 1-edges incident on } v \text{ is odd,} \\ 0 & \text{if the number of 1-edges incident on } v \text{ is even.} \end{cases}$$

For $i \in Z_2$ let $e_f(i) = e(i) = card\{e \in E : f(e) = i\}$ and $v_f(i) = v(i) = card\{v \in V : f^*(v) = i\}$. A labeling f is said to be edge-friendly if $\mid e(0) - e(1) \mid \leq 1$. The 1- edge balance index set ($OEBI$) of a graph G is defined by $\{\mid v_f(0) - v_f(1) \mid :$ the edge labeling f is edge-friendly$\}$. The main purpose of this paper is to completely determine the 1-edge balance index sets of $C_n \times P_3$, $K_{n,n}$.

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1 Introduction

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Varieties of graph labeling have been investigated by many authors [4], [5], [6] and they serve as useful models for broad range of applications.

Recently Chandrashekar Adiga et al. [1, 2, 3] introduced a new labeling called 1-edge balanced labeling and they completely determined the 1-edge balance index sets for $C_n \times P_2$, Chain sum graph of first kind and Double triangular snake graph. The main purpose of this paper is to establish the 1-edge balance index set of some important family of graphs.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and $Z_2 = \{0, 1\}$. Let f be a labeling from $E(G)$ to Z_2, so that the labels of the edges are 0 or 1. The edges labelled 1 are called 1-edges and edges labelled 0 are called 0-edges. The edge labeling f induces a vertex labeling $f^* : V(G) \longrightarrow Z_2$, defined by

$$f^*(v) = \begin{cases} 1 & \text{if the number of 1-edges incident on } v \text{ is odd,} \\ 0 & \text{if the number of 1-edges incident on } v \text{ is even.} \end{cases}$$

For $i \in Z_2$, let $e_f(i) = e(i) = card\{e \in E(G) : f(e) = i\}$ and $v_f(i) = v(i) = card\{v \in V(G) : f^*(v) = i\}$

Definition 1.1. *The graph G is said to be edge-friendly, if $\mid e(1) - e(0) \mid \leq 1$.*

Definition 1.2. *The 1-edge balance index set of a graph G, denoted by $OEBI(G)$, is defined as $\{\mid v_f(1) - v_f(0) \mid : f$ is edge-friendly$\}$.*

For convenience, a vertex is called 0-vertex if its induced vertex label is 0 and 1-vertex, if its induced vertex label is 1.

Before determining the 1-edge balance index sets of $C_n \times P_3$ and $K_{n,n}$ we prove some properties regarding the index numbers and $v_f(1)$.

Theorem 1.3. *If the number of vertices in a graph G is even then the 1-edge balance index set contains only even numbers.*

Proof: Let $G(V, E)$ be a graph with $|V| = n$ which is even. Now suppose that $v_f(1) \geq v_f(0)$ then by the definition of 1- edge balance index set we have

$$v_f(1) + v_f(0) = n$$

$$v_f(1) - v_f(0) = x$$

where x be the element of 1- edge balance index set. Now adding above equations we get

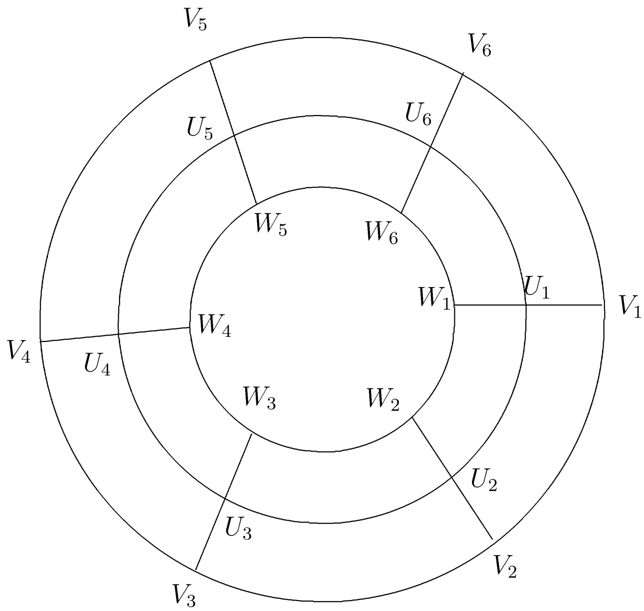
$$2v_f(1) = n + x.$$

Theorem 1.4. *If the number of vertices in a graph G is odd then the 1-edge balance index set contains only odd numbers.*

Theorem 1.5. *In 1-edge balanced labeling, $v_f(1)$ is always even.*

Proof: Let $a_i = |\{v \in V : \text{number of 1-edges incident on } v \text{ is equal to i}\}|$, $i = 1, 2, 3, \ldots n$. Then we have $a_1 + 2a_2 + 3a_3 + \cdots + na_n$ is equal to twice the number of 1-edges, which is even. Therefore $a_1 + 3a_3 + \cdots + na_n$ is even if n is odd and $a_1 + 3a_3 + \cdots + (n-1)a_{n-1}$ is even if n is even. Which implies $a_1 + a_3 + \cdots + a_n = v(1)$ is even if n is odd and $a_1 + a_3 + \cdots + a_{n-1} = v(1)$ is even if n is even.

2 1-edge balance index set of $C_n \times P_3$



$C_n \times P_2$

We describe the problem of finding $OEBI(C_n \times P_3)$ into four cases viz, $n \equiv 0(\mod 4)$, $n \equiv 1(\mod 4)$, $n \equiv 2(\mod 4)$ and $n \equiv 3(\mod 4)$.

Theorem 2.1. *If* $n \equiv 0(\mod 4)$*, then* $OEBI(C_n \times P_3) = \{0, 4, 8, \cdots, 3n\}$.

Proof: Let f be an edge friendly labelling on the graph $C_n \times P_3$. Since $C_n \times P_3$ contains $3n = 3(4k) = 12k$ vertices and $5n = 5(4k) = 20k$ edges, we must have $e(0) = e(1) = 10k$. Denote the vertices of the outer circle of $C_n \times P_3$ by $v_1, v_2, \ldots, v_{4k}$, vertices of the middle circle by $u_1, u_2, \ldots, u_{4k}$ and inner circle by $w_1, w_2, \ldots, w_{4k}$. Now label the edges $u_{4q-2}u_{4q-1}$ for $1 \leq q \leq k$, $u_{4q}u_{4q+1}$ for

$1 \le q \le k-1$, $u_{4k}u_1$ by 1. $v_{4q-3}v_{4q-2}$ for $1 \le q \le k$, $v_{4q}v_{4q+1}$ for $1 \le q \le k-1$, $v_{4k}v_1$ by 1. $w_{4q-3}w_{4q-2}$ for $1 \le q \le k$, $w_{4q}w_{4q+1}$ for $1 \le q \le k-1$, $w_{4k}w_1$ by 1. $v_{2q-1}u_{2q-1}$ for $1 \le q \le 2k$, $u_{2q-1}w_{2q-1}$ for $1 \le q \le 2k$ by 1. Label the remaining edges by 0. Then it is easy to check that $|v(1)-v(0)| = 12k$. Now we interchange the labels of edges $v_{4s-3}v_{4s-2}$ and $v_{4s-2}v_{4s-1}$, for $1 \le s \le k$ we get $|v(0)-v(1)| = 12k-4s = 3n-4s$. Again by interchanging the labels of edges $u_{4s-3}u_{4s-2}$ and $u_{4s-2}u_{4s-1}$ for $1 \le s \le k$ we get $|v(0)-v(1)| = 8k-4s = 2n-4s$. Finally interchanging the labels of $w_{4s-3}w_{4s-2}$ and $w_{4s-2}w_{4s-1}$ for $1 \le s \le k$ we get $|v(0)-v(1)| = 4k-4s$. Thus $\{0, 4, 8, \cdots, 3n\}$ are elements of $OEBI(C_n \times P_3)$.

Since $v(1)$ is always even we have $v(0) = 12k - v(1)$ is also even. Therefore $\{2, 6, 10, \cdots, 3n-2\}$ are not elements of $OEBI(C_n \times P_3)$.

Theorem 2.2. *If* $n \equiv 1(\mod 4)$, *then* $OEBI(C_n \times P_3) = \{1, 3, 5, \ldots, 3n\}$.

Proof: We assume for $k \ge 2$. Let f be an edge friendly labeling on the graph $C_n \times P_3$. Since $C_n \times P_3$ contains $3n = 3(4k+1) = 12k+3$ vertices and $5n = 5(4k+1) = 20k+5$ edges, we have two possibilities: i)$e(0) = 10k+2$ and $e(1) = 10k+3$ or ii) $e(0) = 10k+3$ and $e(1) = 10k+2$.

Case 1: Denote the vertices of the outer circle of $C_n \times P_3$ by $v_1, v_2, \ldots, v_{4k+1}$, middle circle by $u_1, u_2, \ldots, u_{4k+1}$, inner circle by $w_1, w_2, \ldots, w_{4k+1}$. Now label the edges $v_{2q}v_{2q+1}$ for $1 \le q \le 2k$, $u_{3q-2}u_{3q-1}$ for $1 \le q \le k+1$, $w_{q+2}w_{q+3}$ for $1 \le q \le k+1$ and $w_{4k+1}w_1$ by 1. Also $v_{q+1}u_{q+1}$ for $1 \le q \le 4k$, u_1w_1, $u_{3q}w_{3q}$ for $1 \le q \le k+1$ by 1. Label the remaining edges by 0. Then we obtain $v(1) = 0$ and $v(0) = 12k+3$. Therefore $|v(1)-v(0)| = |0-(12k+3)| = 12k+3$. Now by interchanging the labels of edges $v_{4s-3}v_{4s-2}$ and $v_{4s-2}v_{4s-1}$, for $1 \le s \le k$ we get $|v(0)-v(1)| = (12k+3)-4s$. Again by interchanging the labels of the edges $u_{3s-2}u_{3s-1}$ and $u_{3s-1}u_{3s}$ for $1 \le s \le k$ we get $|v(0)-v(1)| = (8k+3)-4s$. Now by interchanging the labels of the edges $w_{3s-1}w_{3s}$ and $w_{3s}w_{3s+1}$ for $1 \le s \le k$

we get $|v(0) - v(1)| = (4k + 3) - 4s$. Therefore $\{3, 7, 10, \cdots, 3n\}$ are elements of $OEBI(C_n \times P_3)$.

Case 2: Label the edges v_1v_2, $v_{4k+1}v_1$, w_1w_2, $w_{4k+1}w_1$, $u_{4k+1}u_1$ by 1. $v_{2q-1}u_{2q-1}$ for $1 \leq q \leq 2k$,$u_{2q+2}w_{2q+2}$ for $1 \leq q \leq 2k - 1$ $u_{2q+1}u_{2q+2}$ for $1 \leq q \leq 2k - 1$,$u_{2q-1}w_{2q-1}$ for $1 \leq q \leq 2k$, $v_{2q+2}u_{2q+2}$ for $1 \leq q \leq 2k - 1$, by 1, label the remaining edges by 0. Now we found that $v(0) = 1$ and $v(1) = 12k+2$. Thus $|v(0) - v(1)| = |1 - (12k + 2)| = 12k + 1$=3n-2

Now by interchanging the labels of the edges $v_{2s-1}v_{2s}$ and $v_{2s}u_{2s}$ for $1 \leq s \leq 2k$,the number of 1- vertices will decrease by two whereas the number of 0-vertices will increase by two and hence the difference $|v(1)-v(0)| = 12k+1-4s = 4k+1$. Again interchanging the labels of the edges $w_{2s}w_{2s+1}$ and $w_{2s+1}u_{2s+1}$ for $1 \leq s \leq k$ we get $|v(1)-v(0)| = 4k+1-4s = 1$. Therefore $\{1, 5, 9, \cdots, 3n-2\}$ are elements of $OEBI(C_n \times P_3)$.

Theorem 2.3. *If* $n \equiv 2(\mod 4)$, *then* $OEBI(C_n \times P_3) = \{2, 6, 10, \ldots, 3n\}$

Proof: Let f be an edge friendly labelling on the graph $C_n \times P_3$. Since $C_n \times P_3$ contains $3n = 3(4k + 2) = 12k + 6$ vertices and $5n = 5(4k + 2) = 20k + 10$ edges, we must have $e(0) = e(1) = 10k + 5$

Denote the vertices of the outer circle of $C_n \times P_3$ by $v_1, v_2, \ldots, v_{4k+2}$, middle circle by $u_1, u_2, \ldots, u_{4k+2}$ and inner circle by $w_1, w_2, \ldots, w_{4k+2}$. Now label the edges $v_{2q-1}v_{2q}$ for $1 \leq q \leq 2k + 1$, u_qu_{q+1} for $1 \leq q \leq 4k + 2$ and u_qw_q for $1 \leq q \leq 4k + 2$ by 1. Label the remaining edges by 0. Then we find that $v(1) = 12k + 6$ and $v(0) = 0$. Therefore $|v(1) - v(0)| = |(12k + 6) - 0| = 3n$.

Now by interchanging the labels of the edges $v_{3s-2}v_{3s-1}$ and $v_{3s-1}v_{3s}$ for $1 \leq s \leq k+1$, we get $|v(1) - v(0)| = 12k + 6 - 4s = 3n - 4s$. Again by interchanging the labels of edges $u_{2s-1}w_{2s-1}$ and $w_{2s-1}w_{2s}$ for $1 \le s \le 2k$ we get $|v(1)-v(0)| = 8k+2-4s$. Thus $\{2,6,10,\dots,3n\}$ are elements of $OEBI(C_n \times P_3)$. Since $v(1)$ is always even we have $v(0) = (12k+6) - v(1)$ is also even. Therefore $\{0,4,8,\dots,3n-2\}$ are not elements of $OEBI(C_n \times P_3)$.

Theorem 2.4. *If* $n \equiv 3(\mod 4)$*, then* $OEBI(C_n \times P_3) = \{1,3,5,\dots,3n\}$

Proof: Let f be an edge friendly labelling on the graph $C_n \times P_3$. Since $C_n \times P_3$ contains $3n = 3(4k+3) = 12k+9$ vertices and $5n = 5(4k+3) = 20k+15$ edges, we must have (i)$e(0) = 10k+7$ and $e(1) = 10k+8$ or (ii)$e(0) = 10k+8$ and $e(1) = 10k+7$

Case 1: Denote the vertices of the outer circle of $C_n \times P_3$ by $v_1, v_2, \dots, v_{4k+3}$, middle circle by $u_1, u_2, \dots, u_{4k+3}$, inner circle by $w_1, w_2, \dots, w_{4k+3}$. Now label the edges $v_{2q}v_{2q+1}$ for $1 \le q \le 2k+1$, u_qu_{q+1} for $1 \le q \le 4k+2$, $u_{4k+3}u_1$,v_1u_1 by 1,u_qw_q for $2 \le q \le 4k+3$,w_1w_2 by 1. Label the remaining edges by "0". Then it is easy to check that $|v(1)-v(0)| = |(12k+8)-1| = 12k+7$. Now we interchange the labels of some edges to get the remaining 1-edge balance index numbers.

By interchanging the labels of the edges $v_{3s-2}v_{3s-1}$ and $v_{3s-1}v_{3s}$ for $1 \le s \le 2k-1$, we get $|(12k+7)-4s| = 4k+11 = n+8$. Again by interchanging the labels $u_{2s}w_{2s}$ and $w_{2s}w_{2s+1}$ for $1 \le s \le k+2$, we get $|v(1)-v(0)| = (4k+11)-4s$. Thus $3,7,9,\dots,3n-2$ are elements of $OEBI(C_n \times P_3)$.

Case 2: Label the edges $v_{4q-1}v_{4q}$ for $1 \le q \le k$ $v_{4q}v_{4q+1}$ for $1 \le q \le k$, $v_{4k+3}v_1$, $u_{4q-1}u_{4q}$ for $1 \le q \le k$,$u_{4q}u_{4q+1}$ for $1 \le q \le k$ by 1. w_qw_{q+1} for $1 \le q \le 4k+2$,$w_{4k+3}w_1$ $u_{4k+3}u_1$,u_qw_q for $1 \le q \le 4k+3$ by 1, label the remaining edges by 0. Now we found that $v(1) = 0$ and $v(0) = 12k+9$. Thus $|v(1)-v(0)| = |0-(12k+9)| = 12k+9$=3n-2

Now by interchanging the labels of the edges $u_{2q-1}w_{2q-1}$ and $w_{2q-1}w_{2q}$ for $1 \leq q \leq 2k+1$, we get $|v(0) - v(1)| = |(12k+9) - 4s|$ the number of 0- vertices will decrease by two whereas the number of 1-vertices will increase by two and hence the difference $|v(0) - v(1)| = 4k+5$. Again interchanging the labels of the edges $v_{4q}v_{4q+1}$ and $v_{4q+1}v_{4q+2}$ for $1 \leq q \leq k$ we get $|v(0) - v(1)| = (4k+5) - 4s$. Interchanging the labels of the edges $v_{4k+3}v_1$ and v_1v_2 we get $|v(0) - v(1)| = 5 - 4 = 1$. Thus $1, 5, 9, \ldots, 3n$ are elements of $OEBI(C_n \times P_3)$ for $n \equiv 3(\mod 4)$.

3 1-edge balance index set of $K_{n,n}$

We describe the problem of finding $OEBI(K_{n,n})$ into four cases viz, $n \equiv 0(\mod 4)$, $n \equiv 1(\mod 4)$, $n \equiv 2(\mod 4)$ and $n \equiv 3(\mod 4)$.

Theorem 3.1. *If $n \equiv 0(\mod 4)$ then $OEBI(K_{n,n}) = \{0, 4, 8, \ldots, 2n\}$,*

Proof: The complete bipartite graph $K_{n,n}$ contains $2n = 2(4k) = 8k$ vertices and $n^2 = (4k)^2 = 16k^2$ edges. Denote the vertices of first part by $v_1, v_2 \ldots v_{4k}$ and second part by $u_1, u_2, \ldots, u_{4k}$. For an edge friendly labeling we must have $e(0) = e(1) = 8k^2$. Now label the edges which are incidence to u_{2q-1} for $1 \leq q \leq 2k$ by 1 and label the remaining edges by 0. Then it is easy to check that $|v(1) - v(0| = |0-(8k)| = 8k$. Now by interchanging the labels of edges $v_{2s-1}u_{2s-1}$ and $v_{2s-1}u_{2s}$ for $1 \leq s \leq 2k$, we get $|v(0)-v(1)| = |8k-4s|$. Thus $\{0, 4, 8, \ldots, 2n\}$ are elements of $OEBI(K_{n,n})$. Since $v(1)$ is always even we have $v(0) = 8k - v(1)$ is also even. Therefore $\{2, 6, 10, \cdots, 2n-2\}$ are not elements of $OEBI(C_n \times P_3)$.

Theorem 3.2. *If $n \equiv 1(\mod 4)$ then $OEBI(K_{n,n}) = \{2, 6, 10, \ldots, 2n\}$*

Proof: Since $K_{n,n}$ contains $2n = 2(4k+1) = 8k+2$ vertices and $n^2 = (4k+1)^2 = 16k^2 + 8k + 1$ edges, denote the vertices of first part by $v_1, v_2 \ldots v_{4k+1}$ and second part by $u_1, u_2, \ldots, u_{4k+1}$. For an edge friendly labeling we must have (i)$e(0) = 8k^2+4k$, $e(1) = 8k^2+4k+1$ or (ii)$e(0) = 8k^2+4k+1$, $e(1) = 8k^2+4k$.

Label the edges u_1v_s for $1 \le s \le 4k+1$, u_sv_{2t-1} for $1 \le s \le 4k-1$, $1 \le t \le 2k+1$ by 1. Label the edge $u_{4k}v_{4k+1}$ and $u_{4k+1}v_{4k+1}$ by 1 and label the remaining edges by 0. Then we have $|v(1) - v(0| = |(8k+2) - 0| = 8k+2$. Now by interchange the labels of the edges $v_{2s}u_{2s}$ and $u_{2s}v_{2s+1}$ for $1 \le s \le 2k$, we get $|v(0) - v(1)| = |(8k+2) - 4s|$. Thus $\{2, 6, 10, \ldots, 2n\}$ are elements of $OEBI(K_{n,n})$. Since $v(1)$ is always even we have $v(0) = 8k+2-v(1)$ is also even. Therefore $\{0, 4, 8, \cdots, 2n-2\}$ are not elements of $OEBI(K_{n,n})$.

Theorem 3.3. *If* $n \equiv 2(\mod 4)$ *then* $OEBI(K_{n,n}) = \{0, 4, 8, \ldots, 2n\}$

Proof: The complete bipartite graph $K_{n,n}$ contains $2n = 2(4k+2) = 8k+4$ vertices and $n^2 = (4k+2)^2 = 16k^2+16k+4$ edges. Denote the first part of vertices by $v_1, v_2, \ldots, v_{4k+2}$ and the second part by $u_1, u_2 \ldots u_{4k+2}$. For edge friendly labelling we must have $e(0) = e(1) = 8k^2+8k+2$. Hence label the edges which are incidence to u_{2q-1} for $1 \le q \le 2k+1$ by 1 and label remaining edges by 0. Then we find that $|v(1) - v(0| = |(4k+2) - (4k+2)| = 0$. Now interchanging the labels of edges $v_{2s-1}u_{2s-1}$ and $v_{2s-1}u_{2s}$ for $1 \le s \le 2k+1$, we get $|v(1) - v(0)| = 4s$. Thus $\{0, 4, 8, \ldots, 2n\}$ are elements of $OEBI(K_{n,n})$.

Theorem 3.4. *If* $n \equiv 3(\mod 4)$ *then* $OEBI(K_{n,n}) = \{2, 6, 10, \ldots, 2n\}$

Proof: Since $K_{n,n}$ contains $2n = 2(4k+3) = 8k+6$ vertices and $n^2 = (4k+3)^2 = 16k^2+24k+9$ edges, denote vertices of first part by $v_1, v_2, \ldots, v_{4k+3}$ and second part by $u_1, u_2, \ldots, u_{4k+3}$. For an edge friendly labeling we must have (i)$e(0) = 8k^2+12k+5$, $e(1) = 8k^2+12k+4$ or (ii)$e(0) = 8k^2+12k+4$, $e(1) = 8k^2+12k+5$.

Now label the edges u_1v_s for $1 \le s \le 4k+2$,$u_{2s+1}v_t$ for $1 \le s \le 2k+1$, $1 \le t \le 4k+2$ by 1 and label remaining edges by 0. Then we get $|v(1) - v(0| = |0 - (8k+6)| = 8k+6 = 2n$. Now by interchanging the labels of the edges

$v_{2s-1}u_{2s-1}$ and $v_{2s-1}u_{2s}$ for $1 \leq s \leq 2k+1$, we get $|v(0) - v(1)| = |(8k+6) - 4s|$. Thus $\{2, 6, 10, \dots, 2n\}$ are elements of $OEBI(K_{n,n})$.

References

[1] Chandrashekar Adiga, C. K. Subbaraya, Shrikanth A. S. and Sriraj M. A., On 1-edge balance index set of some graphs, *Pro. Jang. Math. Soc.*, **14**(2011), 319-331.

[2] Chandrashekar Adiga, Shrikanth A. S. and Shivakumar Swamy C.S., A Note on 1-Edge Balance Index Set, *International J.Math. Combin.*, **3**(2012), 113-117.

[3] Chandrashekar Adiga, Shrikanth A. S., Ying Wang and Yuge Zheng, On 1-edge balance index set of chain sum graphs of first kind, *Pacific-Asian Journal of Mathematics*, **6**(2012), 89-95

[4] L. W. Beineke and S. M. Hegde, Strongly multiplicative graphs, *Discuss. Math. Graph Theory*, 21(2001), 63-76.

[5] I. Cahit, Cordial graphs: A weaker version of graceful and harmonious graphs, *Ars Combin.*, 23(1987), 201-207.

[6] A. Rosa, On certain valuations of vertices of a graph, *Internat. symposium, Rome, July 1976,* Gordon, New York,Breach and Dunod, Paris 1967.

[7] Yuge Zheng and Ying Wang, On the Edge-balance Index sets of $C_n \times P_2$, *2010 International Conference on Networking and Digital Society,* 2(2010)360-363.